Phys514 Fall 2013: Homework 2 Solution

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1 Foot 6.2 (25 pts)

The textbook says that the energy levels correspond to 2S–2P transitions. Therefore, starting from the bottom, the fine levels have $\boxed{L = 0, 1, 1}$ and $\boxed{J = 1/2, 1/2, 3/2}$

The hyperfine splitting in $2P_{3/2}$ satisfies $\Delta E_{hg} < \Delta E_{gf} < \Delta E_{fe}$, which means that F decreases from e to h ($A_{2P_{3/2}}$ is negative). We can calculate the largest F using the interval rule: $F = \frac{\Delta E_{fe}/\Delta E_{gf}}{\Delta E_{fe}/\Delta E_{gf} - 1} = \frac{3/2}{3/2-1} = 3$. Therefore, $\boxed{I = 3/2}$. The following table summarizes the quantum numbers for ^7Li

Label	L	J	F
a	0	1/2	1
b	0	1/2	2
c	1	1/2	1
d	1	1/2	2
e	1	3/2	3
f	1	3/2	2
g	1	3/2	1
h	1	3/2	0

The ratio $A_{2S_{1/2}}/A_{2P_{1/2}}$ is the same in both isotopes because both share the same fine structure. We can calculate X using the h.f.s. of ^6Li as a reference. For ^7Li, $\Delta E_{dc} = 2A_{2S_{1/2}}$ and $\Delta E_{ba} = 2A_{2S_{1/2}}$. For ^6Li, $\Delta E_{dc} = \frac{3}{2}A_{2S_{1/2}}$ and $\Delta E_{ba} = \frac{3}{2}A_{2S_{1/2}}$. Therefore, $X = \left(\frac{803.5\text{MHz}}{228.2\text{MHz}}\right) \cdot 26.1\text{MHz} = \boxed{91.9\text{MHz}}$

2 Hydrogen-Deuterium 1S–2S isotope shift (25 pts)

Mass shift. The n-th energy level has a mass shift of $\Delta E_n \equiv E_n - E_n^\infty = -\frac{hcR_\infty}{n^2}\left(\frac{m_N}{m_e+m_N} - 1\right) \approx \frac{hcR_\infty}{n^2}\frac{m_e}{m_N}$. The mass shift for the 1S–2S transition is given by $\Delta E_{21} \equiv \Delta E_2 - \Delta E_1 = -\frac{3}{4}hcR_\infty\frac{m_e}{m_N}$. Using $m_N = m_p$ for Hydrogen and $m_N = 2m_p$ for Deuterium, we find $\Delta E_{21}^H = -1343.3\text{ GHz}\cdot h$ and $\Delta E_{21}^D = -671.7\text{ GHz}\cdot h$. The isotope shift is $\Delta E_{21}^D - \Delta E_{21}^H = \boxed{671.7\text{ GHz}\cdot h}$

Volume shift. The n-th energy level has a volume shift of[1] $\Delta E_n = hcR_\infty\frac{4Z^4R^2}{5n^3a_0^2}$. The shift associated with the 1S–2S transition is $\Delta E_{21} \equiv \Delta E_2 - \Delta E_1 = -hcR_\infty\frac{7Z^4R^2}{10a_0^2}$. Using that $R = 0.8$ fm for Hydrogen and $R = 2$ fm for Deuterium, we find $\Delta E_{21}^H = -526\text{ kHz}\cdot h$ and $\Delta E_{21}^D = -3290\text{ kHz}\cdot h$. The isotope shift is $\Delta E_{21}^D - \Delta E_{21}^H = \boxed{-2764\text{ kHz}\cdot h}$

The mass shift is in excellent agreement with the value measured, 671 GHz. The volume shift is off by a factor of two from 5234 kHz.

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[1] This is equivalent to the formula seen in class: $\Delta E_n = \frac{e^2}{4\pi\varepsilon_0}\frac{2Z^4R^2}{5n^3a_\mu^3}$, where $a_\mu = a_0\left(1 + \frac{m_e}{m_N}\right) \approx a_0$ and $\frac{e^2}{4\pi\varepsilon_0} = 2a_0hcR_\infty$

3 Zeeman effect in an alkali atom (50 pts)

Part a

The hamiltonian of the hyperfine structure is given by

$$H = A\boldsymbol{I}\cdot\boldsymbol{J} - \boldsymbol{\mu}_J\boldsymbol{B} - \boldsymbol{\mu}_I\boldsymbol{B} \tag{1}$$

where $\boldsymbol{\mu}_J = -g_J\frac{\mu_0}{\hbar}\boldsymbol{J}, \boldsymbol{\mu}_I = g_I\frac{\mu}{\hbar}\boldsymbol{I}$ and $\Delta E_{hfs} = \hbar^2 A(I+1/2)$. As usual, we consider $\boldsymbol{B} = B\hat{z}$. Since we will work in the representation $|m_I, m_J\rangle \equiv |I, m_I\rangle\,|J, m_J\rangle$, it is convenient to express the hamiltonian in terms of ladder operators[2] $I_\pm$ and $J_\pm$.

$$H = \frac{\Delta E_{hfs}}{\hbar^2(2I+1)}(I_+J_- + I_-J_+) + \frac{2\Delta E_{hfs}}{\hbar^2(2I+1)}I_zJ_z + \frac{\mu_0}{\hbar}g_JBJ_z - \frac{\mu_N}{\hbar}g_IBI_z \tag{2}$$

In the particular case of alkali atoms in the ground state, we have $m_J = m_s = \pm 1/2$. The hamiltonian in eq.(2) couples $|m_I, 1/2\rangle$ with $|m_I+1, -1/2\rangle$. In this basis, H takes the matrix form[3]

$$H = \begin{pmatrix} \frac{\Delta E_{hfs}}{2I+1}m_I + \frac{\mu_0}{2}g_JB - \mu_N g_I m_I B & \frac{\Delta E_{hfs}}{2I+1}\sqrt{I(I+1) - m_I(m_I+1)} \\ \frac{\Delta E_{hfs}}{2I+1}\sqrt{I(I+1) - m_I(m_I+1)} & -\frac{\Delta E_{hfs}}{2I+1}(m_I+1) - \frac{\mu_0}{2}g_JB - \mu_N g_I(m_I+1)B \end{pmatrix} \tag{3}$$

The manual diagonalization of H is straightforward but tedious. We can use *Mathematica* instead. The function `Simplify[Eigensystem[H]]` gives the eigenvalues and their respective (unnormalized) eigenvectors. The eigenvalues are

$$\boxed{E_\pm(m_F, B) = -\frac{\Delta E_{hfs}}{2(2I+1)} - \mu_N g_I m_F B \pm \frac{\Delta E_{hfs}x'}{2}} \tag{4}$$

where we have replaced m_I by m_F via $m_I = m_F - 1/2$. The quantities x' and x are defined as

$$x' \equiv \sqrt{1 + \frac{4m_Fx}{2I+1} + x^2} \qquad \text{and} \qquad x \equiv \left(g_J + g_I\frac{\mu_N}{\mu_0}\right)\frac{\mu_0 B}{\Delta E_{hfs}} \tag{5}$$

The eigenvectors associated with $E_\pm$ are [4]

$$\boldsymbol{v}_\pm = \begin{pmatrix} 2m_F + (2I+1)(x \mp x') \\ 2\sqrt{(I+1/2)^2 - m_F^2} \end{pmatrix} \tag{6}$$

From eq.(6) we conclude that the states $|F, m_F\rangle$ are

$$\boxed{|F, m_F\rangle_\pm = \frac{(\boldsymbol{v}_\pm)_1}{\|\boldsymbol{v}_\pm\|}\,|m_F - 1/2, 1/2\rangle + \frac{(\boldsymbol{v}_\pm)_2}{\|\boldsymbol{v}_\pm\|}\,|m_F + 1/2, -1/2\rangle} \tag{7}$$

Part b

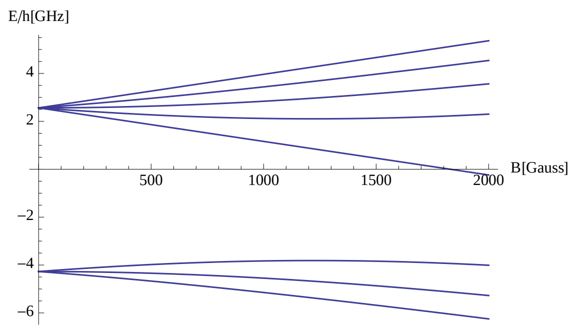
According to Daniel Steck's paper (note that he uses the opposite sign convention for g_I)

ΔE_{hfs}	6.8346826...GHz
g_J	2.002331...
μ_0	$9.274...10^{-24}JT^{-1}$
$g_I\mu_N/\mu_0$	0.000995...

[2] $\boldsymbol{I}\cdot\boldsymbol{J} = I_zJ_z + (I_+J_- + I_-J_+)/2$

[3] The ladder operators $I_\pm$ and $J_\pm$ on the state $|m_I, m_J\rangle$ satisfy $I_\pm\,|m_I, m_J\rangle = \hbar\sqrt{I(I+1) - m_I(m_I \pm 1)}\,|m_I \pm 1, m_J\rangle$ and $J_\pm\,|m_I, \mp 1/2\rangle = \hbar\,|m_I, \pm 1/2\rangle$

[4] The eigenvectors can take equivalent forms since they are not normalized



Part c

We need to calculate the magnetic moment $\mu = -\frac{\partial E}{\partial B}$ and set it to 0. We can use, for example, the function `Derivative[ ]` and then `FindRoot[ ]` on *Mathematica*. For $F = 1, m_F = -1$ we find $\boxed{B = 1221 \text{ Gauss}}$. Plugging this result into eq.(7) gives

$$\boxed{|F = 1, m_F = -1\rangle = \frac{1}{\sqrt{2}}(|-3/2, 1/2\rangle - |-1/2, -1/2\rangle)} \tag{8}$$

We found that $|-3/2, 1/2\rangle$ and $|-1/2, -1/2\rangle$ have equal weights. This is reasonable since they have opposite electron spins (we can ignore the nuclear magnetic moment since $\mu_N \ll \mu_0$).