Phys514 Fall 2013: HW3 Solution

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1 Magnetic resonance (20 pts)

Part a. By identifying the spin state $|\psi_0\rangle = (|+\rangle - |-\rangle)\sqrt{2}$ with the general form $|\psi_0\rangle = \cos(\frac{\theta}{2})|+\rangle + \sin(\frac{\theta}{2})e^{i\varphi}|-\rangle$, we conclude that $\theta = \pi/2$ and $\varphi = \pi$. Therefore $\boldsymbol{S}^{cl} = \frac{\hbar}{2}\sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} = \boxed{-\hbar/2\hat{x}}$

Part b. The oscillating field is on-resonance in all the cases; we will assume that $\gamma < 0$. **(i)** In this case $\boldsymbol{B}_{\text{eff}}$ is parallel to $\boldsymbol{S}^{cl}$, and therefore, the spin $\boldsymbol{S}^{cl}(t) = \boxed{-\hbar/2\hat{x}'}$ remains stationary in the rotating frame. The same behavior would occur if $\boldsymbol{S}^{cl}(t) = \hbar/2\hat{x}'$, which is associated to the state $|\psi_0\rangle = \boxed{(|+\rangle + |-\rangle)/\sqrt{2}}$ **(ii)** In this case $\boldsymbol{B}_{\text{eff}} = -\frac{\gamma|B_\perp}{2}\hat{y}'$. The spin $\boldsymbol{S}^{cl}$ lies in the plane perpendicular to $\hat{y}'$ and rotates with an angular velocity $\omega \equiv \frac{|\gamma|B_\perp}{2}$, i.e. $\boldsymbol{S}^{cl}(t) = \boxed{\hbar/2(-\cos\omega t\,\hat{x}' + \sin\omega t\,\hat{z}')}$. **(iii)** In this case $\boldsymbol{B}_{\text{eff}} = -|\gamma|B_\perp\hat{y}'$ and the spin rotates in the same way as in the part ii, but with twice the angular velocity.

Part c. The wavefunction evolves as

$$|\psi(t)\rangle = e^{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,\frac{\omega t}{2}}|\psi_0\rangle = (\cos\frac{\omega t}{2} - i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\frac{\omega t}{2})|\psi_0\rangle \tag{1}$$

where $\omega \equiv \frac{|\gamma|B_\perp}{2}$ and $\hat{n}$ is opposite to $\boldsymbol{B}_{\text{eff}}$ under the right hand rule convention. **(i)** In this case $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \sigma_x$, then $|\langle -|\psi\rangle|^2 = \boxed{1/2}$. **(ii)** In this case $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \sigma_y$, then$|\langle -|\psi\rangle|^2 = \boxed{1/2(1-\sin\omega t)}$. **(iii)** The same as ii but with twice the angular velocity

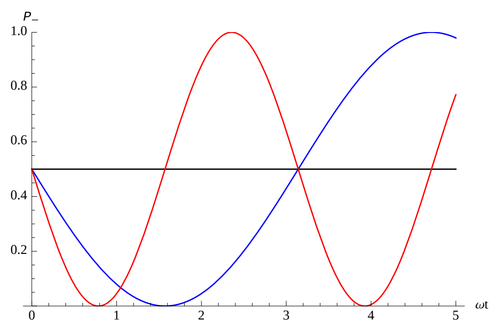

Figure 1: Assuming $\gamma < 0$. Part (i) in black, part (ii) in blue and part (iii) in red.

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2 Beyond the RWA (20 pts)

Part a. The two-level hamiltonian is $H = -\frac{2\mu}{\hbar}\boldsymbol{S}\cdot\boldsymbol{B}$, where $\boldsymbol{B} \equiv B_1\cos(\omega_0 t)\hat{x} + B_0\hat{z}$, $\boldsymbol{S} = \hbar/2(\sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z})$. More explicitly, $H = -\mu B_1 \cos\omega_0 t\ \sigma_x - \mu B_0 \sigma_z$. We look for solutions in the form $|\psi\rangle = a_+(t)\,|+\rangle + a_-(t)\,|-\rangle$. Plugging $|\psi\rangle$ into the Schrödinger equation leads to

$$\boxed{i\frac{d}{dt}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} \omega_0/2 & 2\Omega\cos\omega_0 t \\ 2\Omega\cos\omega_0 t & -\omega_0/2 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}} \tag{2}$$

where $\Omega \equiv -\frac{\mu B_1}{2\hbar}$ and $\omega_0 \equiv -\frac{2\mu B_0}{\hbar}$

Part b. Switching to a rotating frame using $\tilde{a}_+(t) = e^{i\omega_0 t/2}a_+(t)$ and $\tilde{a}_-(t) = e^{-i\omega_0 t/2}a_-(t)$ results in

$$i\frac{d}{dt}\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} = \begin{pmatrix} 0 & \Omega(1+e^{i2\omega_0 t}) \\ \Omega(1+e^{-i2\omega_0 t}) & 0 \end{pmatrix}\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} \tag{3}$$

Under the RWA, the fast oscillatory terms $e^{i2\omega_0 t}$ and $e^{-i2\omega_0 t}$ are averaged out, then

$$i\frac{d}{dt}\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} \tag{4}$$

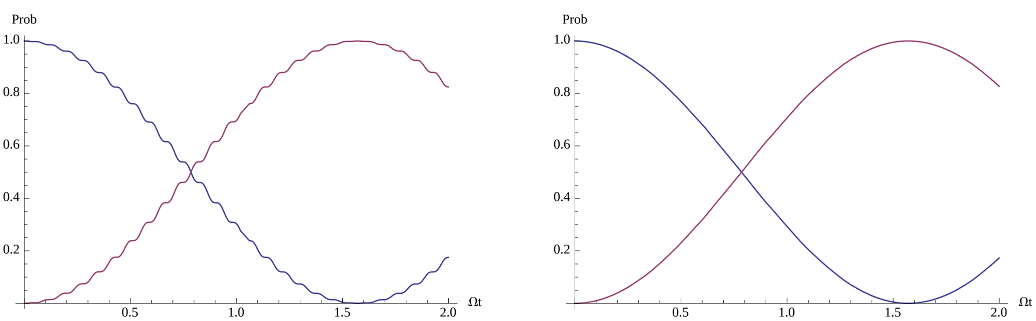
It is clear that the eigenvalues are $\pm\Omega$ and their respective eigenvectors are $\boldsymbol{v}_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$. If we assume that the system has the initial conditions $a_-(0) = \tilde{a}_-(0) = 0$ and $a_+(0) = \tilde{a}_+(0) = 1$, then

$$\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} = \begin{pmatrix} \cos\Omega t \\ -i\sin\Omega t \end{pmatrix} \tag{5}$$

Finally $|\psi(t)\rangle = \boxed{\cos\Omega t\ e^{-i(\Omega+\omega_0/2)t}\,|+\rangle - i\sin\Omega t\ e^{i(\Omega+\omega_0/2)t}\,|-\rangle}$

Part c. For plotting, we can write eq.(3) in a more convenient way

$$i\frac{d}{d(\Omega t)}\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} = \begin{pmatrix} 0 & 1+e^{i8\frac{B_0}{B_1}\Omega t} \\ 1+e^{-i8\frac{B_0}{B_1}\Omega t} & 0 \end{pmatrix}\begin{pmatrix} \tilde{a}_+ \\ \tilde{a}_- \end{pmatrix} \tag{6}$$



(a) $B_1/B_0 = 0.1$ (b) $B_1/B_0 = 0.01$

Figure 2: P_+ in blue and P_- in red. The counter-rotating term appears as fast oscillations.

Mathematica code:

```
α = 10; (*B0/B1*)
eqs = {ⅈ ap'[t] == am[t] (1 + Exp[2 ⅈ 4 α t]),
   ⅈ am'[t] == ap[t] (1 + Exp[-2 ⅈ 4 α t]), am[0] == 0, ap[0] == 1};
sol = NDSolve[eqs, {am, ap}, {t, 2}];
Plot[{Abs[ap[t] /. sol]^2, Abs[am[t] /. sol]^2}, {t, 0, 2}, AxesLabel -> {Ωt, Prob}]
```

3 Matrix elements (20 pts)

The field $\boldsymbol{B}_0 = (1 \text{ Gauss})\hat{z}$ induces a Zeeman splitting $\Delta E = g_F m_F \mu B_0$ on the hyperfine energy levels. On the other hand, $\boldsymbol{B}_1 = B_1 \cos\omega_0 t\ (\hat{x}+\hat{z})$ has both π and $\sigma^{\pm}$ components[1], which couple $|F=3, m_F=0\rangle$ with $\boxed{|4,0\rangle, |4,\pm1\rangle}$, respectively. The frequency detunings associated with those transitions are $\delta_0 \equiv 0$ and $\delta_{\pm1} \equiv \pm g_F \mu B_0/\hbar = \pm(2\pi)\ 0.35$ MHz, respectively[2].

For the Rabi rates, we need to get the matrix elements between the $F=3$ and $F=4$ states. The coupling hamiltonian is $H = g_J \frac{\mu}{\hbar} \boldsymbol{J}\cdot\boldsymbol{B}_1$, where $g_J = 2$ and the nuclear magnetic moment is neglected. We express $|F, m_F\rangle$ in terms of $|m_I, m_J\rangle \equiv |7/2, m_I\rangle\,|1/2, m_J\rangle$ (for example, use the `ClebschGordan` function on Mathematica)

$$
\begin{aligned}
|3,0\rangle &= \tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \tfrac{1}{\sqrt{2}}\left|-\tfrac{1}{2},\tfrac{1}{2}\right\rangle \\
|4,0\rangle &= \tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{2}}\left|-\tfrac{1}{2},\tfrac{1}{2}\right\rangle \\
|4,\pm1\rangle &= \tfrac{1}{2}\sqrt{\tfrac{5}{2}}\left|\pm\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle + \tfrac{1}{2}\sqrt{\tfrac{3}{2}}\left|\pm\tfrac{3}{2},\mp\tfrac{1}{2}\right\rangle
\end{aligned}
$$

The hamiltonian, in terms of ladder operators, is

$$H = \frac{\mu B_1}{\hbar}\left(2J_z + J_+ + J_-\right)\cos\omega_0 t \tag{7}$$

Let us see the effect of those operators on $|3,0\rangle$

$$
\begin{aligned}
J_z\,|3,0\rangle &= -\tfrac{\hbar}{2\sqrt{2}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \tfrac{\hbar}{2\sqrt{2}}\left|-\tfrac{1}{2},\tfrac{1}{2}\right\rangle \\
J_\pm\,|3,0\rangle &= \pm\tfrac{\hbar}{\sqrt{2}}\left|\pm\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle 4,0|\,H\,|3,0\rangle &= -\mu B_1\cos\omega_0 t \\
\langle 4,\pm1|\,H\,|3,0\rangle &= \pm\tfrac{\sqrt{5}\mu B_1}{4}\cos\omega_0 t
\end{aligned}
$$

We conclude that the effective Rabi rates are

$$
\begin{aligned}
(\Omega_{\text{eff}})_0 &= \mu B_1/\hbar = \boxed{(2\pi)\ 14\text{ kHz}} \\
(\Omega_{\text{eff}})_{\pm1} &= \left(\delta^2 + (\sqrt{5}\mu B_1/4\hbar)^2\right)^{1/2} = \boxed{(2\pi)\ 350\text{ kHz}}
\end{aligned}
$$

[1] $\hat{x} = (\hat{e}_{-1} - \hat{e}_1)\sqrt{2}$ and $\hat{z} = \hat{e}_0$

[2] $g_F = +1/4$ for $F=4$; $\mu = h\cdot 1.4$ MHz/Gauss (Steck, Cesium D line data)

4 Quantum projection noise (20 pts)

We will assume $\gamma < 0$. We will use eq.(1) to rotate the spin.

Part a. The initial state is $|\psi_0\rangle \equiv |+\rangle$. Right after the first $\pi/2$-pulse, $|\psi_0'\rangle \equiv e^{-i\sigma_x\pi/4}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$. After an evolution time T, $|\psi_T'\rangle \equiv e^{i\sigma_z T\delta/2}|\psi_0'\rangle = \boxed{1/\sqrt{2}(e^{iT\delta/2}|+\rangle - ie^{-iT\delta/2}|-\rangle)}$

Part b.

(i) After the second $\pi/2$-pulse $|\psi_T''\rangle \equiv e^{-i\sigma_x\pi/4}|\psi_T'\rangle = i\sin T\delta/2\,|+\rangle - i\cos T\delta/2\,|-\rangle$. If $T\delta = \pm\pi/2$, then $|\psi_T''\rangle_\pm = \boxed{i/\sqrt{2}(\pm|+\rangle - |-\rangle)}$

(ii) If we consider an arbitrary state $|\psi\rangle \equiv \alpha|+\rangle + \beta|-\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$, then

$$\langle S_z\rangle = \frac{\hbar}{2}(|\alpha|^2 - |\beta|^2) = \frac{\hbar}{2}(1 - 2|\beta|^2) = \boxed{\hbar/2(1 - 2P_-)} \tag{8}$$

(iii) $\Delta S_z = (\langle S_z^2\rangle - \langle S_z\rangle^2)^{1/2} = \hbar/2$. Then, $P_- = \frac{1}{2}(1 - \frac{\langle S_z\rangle}{\hbar/2})$ implies $\Delta P_- = -\frac{1}{\hbar}\Delta\langle S_z\rangle$. Therefore $\Delta_- \equiv |\Delta P_-| = \boxed{1/2}$

Part c. The uncertainty in ω_0 is given by $\Delta\omega_0 = 1/2\sqrt{\Delta\omega_u^2 + \Delta\omega_l^2}$. To calculate $\Delta\omega_u$ and $\Delta\omega_l$ in terms of $\Delta P_- = \frac{\Delta_-}{\sqrt{N}}$, we use the derivative of $P_- \equiv |\langle -|\psi_T''\rangle|^2 = \frac{1}{2} + \frac{1}{2}\cos(\omega - \omega_0)T$ with respect to ω, evaluated at ω_u and ω_l. In both cases $\left|\frac{dP_-}{d\omega}\right| = \frac{T}{2}$, therefore $\Delta\omega_u = \Delta\omega_l = \frac{2\Delta_-}{T\sqrt{N}}$. Finally, the uncertainty in ω_0 is $\Delta\omega_0 = \frac{2\sqrt{2}\Delta_-}{T\sqrt{N}} = \boxed{1/(T\sqrt{2N})}$

5 Spin echo (20 pts)

We will assume $\gamma = 2\mu/\hbar < 0$

Part a. Right after the first $\pi/2$-pulse, the spin points towards $-\hat{y}'$. As time evolves, the spin rotates about $\hat{z}'$ towards $-\hat{x}'$ if $\delta B_0 > 0$ and towards $\hat{x}'$ if $\delta B_0 < 0$. Therefore, after a time T, the spin will be uniformly distributed at angles $\boxed{\theta \in [-\theta_m, \theta_m]}$ with respect to $-\hat{y}'$, where $\theta_m \equiv \frac{|\mu|B_0 T}{50\hbar}$

Part b. The second $\pi/2$-pulse rotates the spin about $+\hat{x}'$ by $\pi/2$. In the last problem, we found that $P_- = 1/2 + 1/2\cos T\delta$. Therefore $\boxed{P_- \in [1/2 + 1/2\cos\theta_m, 1]}$. For $\langle S_z\rangle$, we use eq.(8), then $\boxed{\langle S_z\rangle \in [-\hbar/2, -\hbar/2\cos\theta_m]}$

To calculate $\langle P_-\rangle_N$, we consider the angular probability density $= 1/(2\theta_m)$, therefore

$$\langle P_-\rangle_N = \int_{-\theta_m}^{\theta_m}\left(\frac{1}{2} + \frac{1}{2}\cos\theta\right)\frac{1}{2\theta_m}d\theta = \boxed{1/2 + 1/(2\theta_m)\,\sin\theta_m}$$

Part c. Let us take the particular case $\delta B_0^* > 0$. Right before the π-pulse, the spin is in the x'–y' plane at an angle $\theta^* = 2|\mu|\delta B_0^* T/(2\hbar)$ with respect to $-\hat{y}'$ (measured clockwise) . The π-pulse rotates the spin about $\hat{x}'$ by π, i.e. now the spin is at θ^* with respect to $+\hat{y}'$ (measured counterclockwise) . After an evolution time $T/2$, the spin becomes aligned with $\hat{y}'$, which is independent of δB_0. Finally, after the second $\pi/2$-pulse, the spin returns to the initial state $|+\rangle$. Therefore, $\boxed{P_- = 0}$, $\boxed{\langle S_z\rangle = \hbar/2}$ for every run, and $\boxed{\langle P_-\rangle_N = 0}$