Phys514 Fall 2013: HW4 Solution

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1 Von Neumann entropy (25 pts)

a. Pure state: the eigenvalues of $\rho = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ are $\lambda = 0, 1$, therefore $S = \ln 1 = \boxed{0}$

b. Mixed state: $S = -2 \cdot 1/2 \ln 1/2 = \boxed{\ln 2}$

c. Entangled state: if we consider the basis ordering $++, +-, -+, --$, then

$$\rho = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1/2 & -1/2 & 0\\ 0 & -1/2 & 1/2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad \rho_a = \rho_b = \begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}$$

The eigenvalues of ρ are 0 and 1, then $S = \ln 1 = \boxed{0}$. Also $S_a = S_b = -2 \cdot 1/2 \ln 1/2 = \boxed{\ln 2}$

d. Non-entangled state:

$$\rho = \begin{pmatrix}1/4 & -1/4 & 1/4 & -1/4\\ -1/4 & 1/4 & -1/4 & 1/4\\ 1/4 & -1/4 & 1/4 & -1/4\\ -1/4 & 1/4 & -1/4 & 1/4\end{pmatrix} \quad \rho_a = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix} \quad \rho_b = \begin{pmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{pmatrix}$$

The eigenvalues of ρ, ρ_a and ρ_b are 0 and 1, therefore, $S = S_a = S_b = \ln 1 = \boxed{0}$

From the examples above we could infer that $S = 0$ for a pure state and $S > 0$ for a mixed state. Also, the entropy of the reduced density matrices is zero if the state is not entangled; and greater than zero if it is entangled.

2 Foot 1.8 (25 pts)

From Foot 1.23, $E(t) = E_0 e^{-t/\tau}$, where $E_0 = \frac{1}{2} m\omega^2 r^2$ and $\tau = \frac{6\pi\epsilon_0 mc^3}{e^2\omega^2}$. The transition rate is $t^{-1} = \frac{1}{\tau}\ln\left(1 - \frac{\hbar\omega}{E_0}\right) \approx \frac{E_0}{\hbar\omega\tau} = \frac{e^2 r^2\omega^3}{12\pi\epsilon_0\hbar c^3}$. Plugging numbers in and using that $r = n^2 a_0$ (we treat this problem classically) gives $\tau = 19.35$ ns and $E_0/(\hbar\omega) = 2.81$ and $t^{-1} = \boxed{1.45 \cdot 10^8\ \mathrm{s}^{-1}}$

3 Foot 7.5 (25 pts)

a. From Foot 7.81, the on-resonance cross-section is $\sigma(\omega_0) = \frac{3}{2\pi}\lambda_0^2 = 1.66 \cdot 10^{-13}\ \mathrm{m}^2$. From Foot 7.70, $I = I_0 e^{-n\sigma(\omega_0)\Delta z} = I_0 e^{-1}$, then $n = \frac{1}{\sigma(\omega_0)\Delta z} = 6.04 \cdot 10^{15}\ \mathrm{m}^{-3}$. Therefore, $N = \frac{4}{3}\pi(0.5\ \mathrm{mm})^3 n = \boxed{3.16 \cdot 10^6}$

b. From Foot 7.84, the absorption coefficient is $\kappa(\omega_0, I_{\mathrm{sat}}) = \frac{n\sigma(\omega_0)}{2} = 0.5\ \mathrm{mm}^{-1}$, then the absorption is $1 - e^{-n\sigma(\omega_0)\Delta z} = \boxed{0.39}$

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4 Foot 7.6 (25 pts)

a. The electric dipole selection rule $\Delta l = \pm 1$ allows the following transitions: $2p - 3s$, $1s - 3p$, $2s - 3p$, $2p - 3d$ and $1s - 2p$

b. For $3s$, $\tau = 1/(6.3 \cdot 10^6) = \boxed{158.73 \text{ ns}}$. For $3p$, $\tau = 1/(1.7 \cdot 10^8) + 1/(2.2 \cdot 10^7) = \boxed{5.21 \text{ ns}}$. For $3d$, $\tau = 1/(6.5 \cdot 10^7) = \boxed{15.38 \text{ ns}}$. The fraction in the $2s$ configuration will be $\frac{N_{2s}}{N_{2s}+N_{1s}} = \left(1 + \frac{A_{1s\text{-}3p}}{A_{2s\text{-}3p}}\right)^{-1} = \boxed{0.11}$. The relation $N_{1s}/N_{2s} = A_{1s\text{-}3p}/A_{2s\text{-}3p}$ was used.

c. For simplicity, let us ignore the decay process $3p \to 2s$, which is dominated by $3p \to 1s$. The lifetime $\tau = 1/A$ is proportional to $1/(\omega^3 |D|^2)$. On the one hand, the wavefunctions have a greater overlap in the $1s - 2p$ configuration than in $1s - 3p$, therefore $D_{1s\text{-}2p} > D_{1s\text{-}3p}$. On the other hand, $\omega_{1s\text{-}2p} < \omega_{1s\text{-}3p}$. To determine which one has a bigger effect, we use the table shown in the part d.

$$\frac{\tau_{1s\text{-}2p}}{\tau_{1s\text{-}3p}} \approx \left(\frac{\omega_{1s\text{-}3p}}{\omega_{1s\text{-}2p}}\right)^3 \left(\frac{D_{1s\text{-}3p}}{D_{1s\text{-}2p}}\right)^2 = 0.27$$

d. Using Foot 7.23, $A_{ij} = \frac{g_i}{g_j} \frac{4\alpha}{3c^2} \omega_{ij}^3 |D_{ji}|^2$, where α is the fine structure constant and $\omega_{ij} = 2\pi c R_\infty \left(\frac{1}{n_j^2} - \frac{1}{n_i^2}\right)$.

$i - j$	$A_{ij}/10^6$[1/s]	g_i/g_j	$\omega/10^{15}$[rad/s]	$D_{ij}[a_0]$
$2p - 3s$	6.3	3	2.87	0.54
$1s - 3p$	170	1/3	18.4	0.52
$2s - 3p$	22	1/3	2.87	3.03
$2p - 3d$	65	3/5	2.87	3.89
$1s - 2p$	625	1/3	15.5	1.29

e. Using the formula $I_{sat} = \frac{\pi h c}{3\lambda^3 \tau}$

Transition	τ[ns]	I_{sat}[W/m^2]
$2p - 3s$	158.73	4.66
$1s - 3p$	5.88	32835