HW10 solution[1] - Phys487 Spring 2015

Problem 1 (3 points)

By definition $T = (e^{2i\delta_0} + e^{2i\delta_1})/2$ and $R = (e^{2i\delta_0} - e^{2i\delta_1})/2$ (lecture note 22 page 8). Therefore,

$$\begin{aligned} T &= e^{i(\delta_0+\delta_1)} \frac{e^{i(\delta_0-\delta_1)} + e^{-i(\delta_0-\delta_1)}}{2} \\ &= e^{i(\delta_0+\delta_1)} \cos(\delta_0 - \delta_1) \end{aligned} \tag{1}$$

and

$$\begin{aligned} R &= ie^{i(\delta_0+\delta_1)} \frac{e^{i(\delta_0-\delta_1)} - e^{-i(\delta_0-\delta_1)}}{2i} \\ &= ie^{i(\delta_0+\delta_1)} \sin(\delta_0 - \delta_1) \end{aligned} \tag{2}$$

Problem 2 (6 points)

We consider a square well (attractive potential) with

$$V(x) = \begin{cases} 0 & \text{if } |x| \geq L/2 \\ -V_0 & \text{if } |x| \leq L/2 \end{cases} \tag{3}$$

with $V_0 > 0$. Because the potential is symmetric about 0, we will only consider $x \geq 0$. The *even* solution is

$$\psi(x) = \begin{cases} A_0 \cos(k'x) & \text{if } x \leq a \\ B_0 \cos(kx + \delta_0) & \text{if } x \geq a \end{cases} \tag{4}$$

where $a = L/2$, $k = \sqrt{2mE}/\hbar$ and $k' = \sqrt{2m(E+V_0)}\hbar$. The boundary conditions at $x = a$ are $\psi(a^+) = \psi(a^-)$ and $\psi'(a^+) = \psi'(a^-)$, which lead to

$$\begin{aligned} k \tan(ka + \delta_0) &= k' \tan k'a \\ \frac{\tan ka + \tan \delta_0}{1 - \tan ka \tan \delta_0} &= \frac{k'}{k} \tan k'a. \end{aligned} \tag{5}$$

Solving for $\tan \delta_0$ results in

$$\tan \delta_0 = \frac{\frac{k'}{k} \tan k'a - \tan ka}{1 + \frac{k'}{k} \tan k'a \tan ka}. \tag{6}$$

Analogously, the *odd* solution is

$$\psi(x) = \begin{cases} A_1 \sin(k'x) & \text{if } x \leq a \\ B_1 \sin(kx + \delta_1) & \text{if } x \geq a \end{cases} \tag{7}$$

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and the boundary conditions at $x = a$ lead to

$$k' \tan(ka + \delta_1) = k \tan k'a$$

$$\tan \delta_1 = \frac{\frac{k}{k'} \tan k'a - \tan ka}{1 + \frac{k}{k'} \tan k'a \tan ka} \tag{8}$$

The resonance condition is $\cot \delta_0 - \cot \delta_1 = 0$ (lecture 24, page 3), and therefore,

$$\cot \delta_0 - \cot \delta_1 = \frac{\left(\frac{k}{k'} - \frac{k'}{k}\right) \tan k'a \,(1 + \tan^2 ka)}{\left[\frac{k'}{k} \tan k'a - \tan ka\right] \left[\frac{k}{k'} \tan k'a - \tan ka\right]} = 0 \tag{9}$$

which is zero when $\tan k'a = 0$ and also when $\tan k'a = \infty$. Thus, resonances occur when $k'a = n\pi/2$ ($k' = \sqrt{2m(E + V_0)}\hbar$ must be real, so $E \geq -V_0$) or, equivalently,

$$\boxed{E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2} - V_0}. \tag{10}$$

This is the same result as Griffiths 2.171.

The condition for bound states is $\cot \delta_l - i = 0$ (lecture 24, page 5). For even states, Eq. (6) leads to

$$(1 + i \tan ka) \left(1 - \frac{ik'}{k} \tan k'a\right) = 0. \tag{11}$$

To have bound states, $k = \sqrt{2mE}/\hbar$ must be imaginary, so necessarily $E < 0$. Defining $k_b = \sqrt{-2mE}/\hbar$ and using the identity $\tan ik_b a = i \tanh k_b a$, we find

$$(1 - \tanh k_b a) \left(1 - \frac{k'}{k_b} \tan k'a\right) = 0. \tag{12}$$

The first term in parentheses is never zero for finite E. The second term gives the transcendental equation $\boxed{\tan k'a = k_b/k'}$ for even bound states.

For odd states, Eq. (8) leads to

$$(1 + i \tan ka) \left(1 - \frac{ik}{k'} \tan k'a\right) = 0$$

$$(1 - \tan k_b a) \left(1 + \frac{k_b}{k'} \tan k'a\right) = 0 \tag{13}$$

and the second parentheses yield the equation $\boxed{\tan k'a = -k'/k_b}$ for odd bound states.

We have concluded that bound states are allowed only when $E < 0$. In the case of a repulsive potential in the form

$$V(x) = \begin{cases} V_0 & \text{if } |x| \geq L/2 \\ 0 & \text{if } |x| \leq L/2 \end{cases} \tag{14}$$

with $V_0 > 0$, bound states are forbidden because E can not be below $V_{\min} = 0$.

Problem 11.11 (3 points)

$$
\begin{aligned}
f(\theta) &= \frac{2m\beta}{\hbar^2\kappa}\int_0^\infty e^{-\mu r}\sin(\kappa r)dr \\
&= \frac{2m\beta}{\hbar^2\kappa}\frac{1}{2i}\int_0^\infty \left[e^{-(\mu-i\kappa)r} - e^{-(\mu+i\kappa)r}\right]dr \\
&= \frac{2m\beta}{\hbar^2\kappa}\frac{1}{2i}\left[\frac{1}{\mu-i\kappa} - \frac{1}{\mu+i\kappa}\right] \\
&= -\frac{2m\beta}{\hbar^2(\mu^2+\kappa^2)}
\end{aligned} \tag{15}
$$

Problem 11.12 (3 points)

Eq. (15) and $\kappa = 2k\sin(\theta/2)$ (Griffiths 11.89) imply

$$
\begin{aligned}
\sigma &= \int |f(\theta)|^2 \sin\theta d\theta d\phi \\
&= \frac{2\pi}{\mu^4}\left(\frac{2m\beta}{\hbar^2}\right)^2 \int_0^\pi \frac{\sin\theta d\theta}{\left[1+(2k/\mu)^2\sin^2\theta/2\right]^2}
\end{aligned} \tag{16}
$$

We use 'Simplify[Integrate[sin[θ]/(1+α^2 sin^2[θ/2])^2, {θ, 0, π}], {$\alpha \in$ Reals, $\alpha > 0$}]' on Mathematica to calculate the integral, which results in $1/[1+(2k/\mu)^2]$. Therefore

$$
\sigma = \pi\left(\frac{4m\beta}{\mu\hbar^2}\right)^2\frac{1}{\mu^2+4k^2} \tag{17}
$$

where $k^2 = 2mE/\hbar^2$.