HW3 solution[1] - Phys487 Spring 2015

Problem 5.18 (6 points)

(a) The general solution is $\psi = A\sin kx + B\cos kx$. From Griffiths 5.63 we have $A\sin ka = (e^{iKa} - \cos ka)B$. Therefore,

$$\begin{aligned}\psi &= A\sin kx + \frac{A\sin ka}{e^{iKa} - \cos ka}\cos kx \\ &= \frac{A}{e^{iKa} - \cos ka}\left[e^{iKa}\sin kx - \sin kx\cos ka + \cos kx\sin ka\right] \\ &= C\left[\sin kx + e^{-iKa}\sin k(a-x)\right],\end{aligned}$$

where $C = Ae^{iKa}/(e^{iKa} - \cos ka)$.
(b) If $z = ka = j\pi$, then $\sin ka = 0$ and Griffiths 5.64 implies that $\cos Ka = \cos ka = (-1)^j$. Therefore, $\sin Ka = 0$ and $e^{iKa} = \cos Ka + i\sin Ka = (-1)^j$. Using this in 5.62 gives

$$kA - (-1)^j k\left[A(-1)^j - 0\right] = \frac{2m\alpha}{\hbar^2}B$$

Therefore $B = 0$ and $\psi = A\sin kx$, this is, ψ is zero at each delta spike.

Problem 5.20 (6 points)

The positive-energy solution is the same:

$$\cos Ka = \cos ka + \frac{m\alpha}{\hbar^2 k}\sin ka \tag{1}$$

with $\alpha < 0$, $k = \sqrt{2mE}/\hbar$, $E > 0$ and $K = 2\pi n/Na$ (with n integer).

For the negative-energy solution, the wavevector is now defined as $\tilde{k} = \sqrt{-2mE}/\hbar$, since $E < 0$ and $\tilde{k}$ has to be real. Notice that $\tilde{k} = ik \Rightarrow k = -i\tilde{k}$. Then, we can substitute k into Eq. (1)

$$\begin{aligned}\cos Ka &= \cos(-i\tilde{k}a) + \frac{m\alpha}{\hbar^2(-i\tilde{k})}\sin(-i\tilde{k}a) \\ &= \cosh\tilde{k}a + \frac{m\alpha}{\hbar^2\tilde{k}}\sinh\tilde{k}a \end{aligned}\tag{2}$$

where we have used that $\cos(ix) = \cosh x$ and $\sin(ix) = i\sinh x$.

In order to plot Eqs. (1) and (2) together, we define $z = ka > 0$ for the positive-energy solution, $\cos ka + \beta\sin z/z$, and $z = -\tilde{k}a < 0$ for the negative-energy solution, $\cosh z + \beta\sinh z/z$. Therefore

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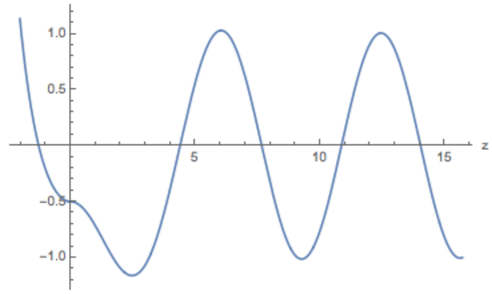

The first band is partly positive and partly negative and contains N states because $\cos Ka = \cos 2\pi n/N$.

Problem 5.25 (3 points)

- For $N = 1$, the ball can go in any of the d baskets, so d ways.
- For $N = 2$, $d + d(d-1)/2 = d(d+1)/2$ ways, because
  - 2 balls in the same basket: d
  - First ball in any of the d baskets; second ball in any of the $d-1$ baskets left. Balls are indistinguishable, then we divide by 2. So, $d(d-1)/2$ ways.
- For $N = 3$, $d + d(d-1) + d(d-1)(d-2)/3! = d(d+1)(d+2)/3!$, because
  - 3 balls in the same basket: d
  - 2 in one basket, 1 in another: $d(d-1)$
  - each ball in a different basket: $d(d-1)(d-2)/3!$
- For $N = 4$, $d(d+1)(d+2)(d+3)/4!$, because
  - 4 balls in the same basket: d
  - 3 in one, 1 in another: $d(d-1)$
  - 2 in one, 2 in another: $d(d-1)/2$
  - 2 in one, 1 each in another: $d(d-1)(d-2)/2$
  - each ball in a different basket: $d(d-1)(d-2)(d-3)/4!$

The general formula seems to be $\binom{d+N-1}{N}$. The validity of this formula has to be proved by induction. However, the problem was graded up to this point.

Problem 5.29 (12 points)

(a) The Bose-Einstein distribution $1/(e^{(\epsilon-\mu)/k_BT}-1) > 0$ has to be positive. Therefore $e^{(\epsilon-\mu)/k_BT} > 1$ and then $\epsilon > \mu$ for all allowed energies ϵ.
(b) The ground energy of a free particle is (approximately) zero. Therefore, part (a) implies that $\mu \leq 0$. Griffiths 5.108 is

$$\frac{N}{V} = \frac{1}{2\pi^2}\int_0^\infty \frac{k^2}{e^{(\hbar^2k^2/2m-\mu)/k_BT}-1}dk.$$

In order to hold N/V constant as T decreases, $\hbar^2 k^2/2m - \mu$ must also decrease, which implies that μ must increase (up to zero, since $\mu \le 0$).
(c) Under the change of variables $x = \hbar^2 k^2/2mk_BT$, the equation in part (b) with $\mu = 0$ becomes

$$\frac{N}{V} = \frac{1}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2}\Gamma(3/2)\zeta(3/2)$$

where we have used $\int_0^\infty x^{1/2}/(e^x - 1)dx = \Gamma(3/2)\zeta(3/2)$. Using that $\Gamma(3/2) = \sqrt{\pi}/2$ and $\zeta(3/2) \approx 2.6$, we conclude that

$$T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{2.6V}\right)^{2/3}$$

(d)

$$\frac{N}{V} = \frac{\text{mass/volume}}{\text{mass/atom}} = \frac{0.15 \cdot 10^3 \text{ kg/m}^3}{4(1.67 \cdot 10^{-27} \text{ kg})} = 2.2 \cdot 10^{28} \text{ m}^{-3}$$

$$T_c = \frac{2\pi(1.05 \cdot 10^{-34} \text{ J} \cdot s)^2}{4(1.67 \cdot 10^{-27} \text{ kg})(1.38 \cdot 10^{-23} \text{ J/K})}\left(\frac{2.2 \cdot 10^{28}}{2.61 \text{ m}^3}\right)^{2/3} = 3.1 \text{ K}$$

Problem 5.33 (3 points)

(a) Each particle has 3 possible states, therefore, there are $3 \times 3 \times 3 = 27$ different three-particle states.
(b) There are 10 possible combinations: aaa, bbb, ccc, aab, aac, bba, bbc, cca, ccb, abc.
(c) Only one state is allowed: abc.

Problem 5.37 (9 points)

(a) The energy of a 1D harmonic oscillator is $\epsilon_{1D} = \hbar\omega(n + 1/2)$. In 3D, the problem is separable in each coordinate, then the total energy is $\epsilon_{3D} = \hbar\omega(n_x + n_y + n_z + 3/2)$.

The total particle number is given by

$$N = \sum_{n_x,n_y,n_z=0}^{\infty} e^{-(\epsilon_{3D}-\mu)/k_BT} = e^{(\mu-3\hbar\omega/2)/k_BT}\left(\sum_n^{\infty} e^{-n\hbar\omega/k_BT}\right)^3 = e^{(\mu-3\hbar\omega/2)/k_BT}\left(\frac{1}{1 - e^{-\hbar\omega/k_BT}}\right)^3$$

Therefore, $\mu = k_BT[\ln N + 3\hbar\omega/(2k_BT) + 3\ln(1 - e^{-\hbar\omega/k_BT})]$

The total energy is given by

$$\begin{aligned} E &= \sum_{n_x,n_y,n_z=0}^{\infty} \epsilon_{3D} e^{-(\epsilon_{3D}-\mu)/k_BT} \\ &= \hbar\omega e^{(\mu-3\hbar\omega/2)/k_BT} \underbrace{\sum_{n_x,n_y,n_z=0}^{\infty} (n_x + n_y + n_z + 3/2)\, x^{n_x+n_y+n_z}}_{\frac{3(1+x)}{2(1-x)^4}} \end{aligned}$$

where $x = e^{-\hbar\omega/k_BT}$. The sum was evaluated using Mathematica: Sum[(nx + ny + nz + 3/2) x∧(nx + ny + nz), {nx, 0, ∞ }, {ny, 0, ∞}, {nz, 0, ∞}].

Then, using that $N = e^{(\mu - 3\hbar\omega/2)/k_B T}(1-x)^{-3}$, we conclude that $E = \frac{3}{2} N\hbar\omega \frac{1+e^{-\hbar\omega/k_B T}}{1-e^{-\hbar\omega/k_B T}}$.

(b) When $k_B T \ll \hbar\omega$ (low temperature), $e^{-\hbar\omega/k_B T} \approx 0$, and therefore, $E \approx \frac{3}{2} N\hbar\omega$. This result corresponds to $n_x, n_y, n_z \approx 0$ for all particles.

(c) When $k_B T \gg \hbar\omega$ (high temperature), we expand $e^{-\hbar\omega/k_B T} \approx 1 - \hbar\omega/k_B T$, and therefore, $E \approx 3Nk_B T$. This result agrees with the equipartition theorem for a classical gas: each degree of freedom contributes with $k_B T/2$. In this case, there are 6 degrees of freedom (3 kinetic and 3 potential) and N particles.

Problem 2 (3 points)

The hamiltonian is

$$H = -\sum_n \Big[t_1 |n\rangle\langle n+1| + t_2 |n\rangle\langle n+2| \Big] + \text{h.c.}$$

Using $|n\rangle = \sum_k e^{ikna}|k\rangle$, the hamiltonian becomes

$$H = -\sum_{n,k,k'} e^{i(k-k')na} \Big[t_1 e^{-ik'a} + t_2 e^{-i2k'a} \Big] |k\rangle\langle k'| + \text{h.c.}$$

but $\sum_n e^{i(k-k')na} = \delta_{k,k'}$, then

$$\begin{aligned} H &= -\sum_k \Big[t_1 e^{-ika} + t_2 e^{-i2ka} \Big] |k\rangle\langle k| + \text{h.c.} \\ &= -\sum_k \Big[2t_1 \cos(ka) + 2t_2 \cos(2ka) \Big] |k\rangle\langle k| \end{aligned}$$

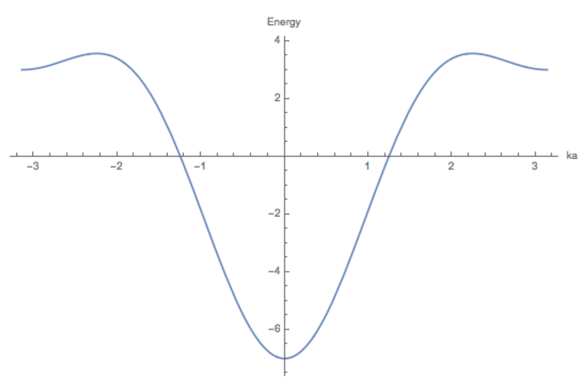

Figure 1: $t_1 = 5$ and $t_2 = 2$