HW8 solution[1] - Phys487 Spring 2015

Problem 9.7 (9 points)

(a) The Schrödinger equation leads to

$$\dot{c}_a = -\frac{iV_{ab}}{2\hbar} e^{i(\omega-\omega_0)t} c_b \tag{1}$$

$$\dot{c}_b = -\frac{iV_{ba}}{2\hbar} e^{-i(\omega-\omega_0)t} c_a. \tag{2}$$

Solving for c_b yields

$$\ddot{c}_b + i(\omega-\omega_0)\dot{c}_b + \frac{|V_{ab}|^2}{4\hbar^2} c_b = 0. \tag{3}$$

Using the ansatz $e^{\lambda t}$, we find the eingenvalues

$$\lambda = -\frac{i(\omega-\omega_0)}{2} \pm i\omega_r, \tag{4}$$

where

$$\omega_r = \frac{1}{2}\sqrt{(\omega-\omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}}. \tag{5}$$

The solution for $c_a(t)$ and $c_b(t)$ with the initial conditions $c_a(0) = 1$ and $c_b(0) = 0$ is, respectively

$$\begin{aligned} c_a(t) &= e^{i(\omega-\omega_0)t/2}\left[\cos\omega_r t - i\left(\frac{\omega-\omega_0}{2\omega_r}\right)\sin\omega_r t\right] \\ c_b(t) &= -\frac{i}{2\hbar\omega_r} V_{ba} e^{-i(\omega-\omega_0)t/2}\sin\omega_r t. \end{aligned} \tag{6}$$

(b) From the last result we have

$$P_{a\to b}(t) = |c_b(t)|^2 = \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2}\sin^2\omega_r t \le \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2}. \tag{7}$$

From Eq. (5), it is evident that $P_{a\to b}(t)$ is less than or equal to 1. The total probability is normalized to 1, since

$$\begin{aligned} |c_a|^2 + |c_b|^2 &= \cos^2\omega_r t + \left(\frac{\omega-\omega_0}{2\omega_r}\right)^2 \sin^2\omega_r t + \frac{|V_{ab}|^2}{4\hbar^2\omega_r^2}\sin^2\omega_r t \\ &= \cos^2\omega_r t + \sin^2\omega_r t \\ &= 1 \end{aligned} \tag{8}$$

(c,d) If $|V_{ba}|/\hbar \ll |\omega-\omega_0|$, then $\omega_r \approx |\omega-\omega_0|/2$ and

$$P_{a\to b} \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2(\omega-\omega_0)\,t/2}{(\omega-\omega_0)^2}, \tag{9}$$

which confirms Eq. 9.28. The system first returns to its initial state at $t = \pi/\omega_r$.

---

[1] Grader: David Chen, dchen30@illinois.edu

Problem 9.12 (6 points)

$[L^2, z] = [L_x^2 + L_y^2 + L_z^2, z] = L_x[L_x, z] + [L_x, z]L_x + L_y[L_y, z] + [L_y, z]L_y + L_z[L_z, z] + [L_z, z]L_z$. But

$$\begin{aligned} [L_x, z] &= [yp_z - zp_y, z] = y[p_z, z] = -i\hbar y \\ [L_y, z] &= [zp_x - xp_z, z] = -x[p_z, z] = i\hbar y \\ [L_z, z] &= [xp_y - yp_x, z] = 0 \end{aligned} \tag{10}$$

Then, $[L^2, z] = i\hbar(-L_x y - yL_x + L_y x + xL_y)$. But

$$\begin{aligned} L_x y &= [L_x, y] + yL_x = i\hbar z + yL_x \\ L_y x &= [L_y, x] + xL_y = -i\hbar z + xL_y \end{aligned} \tag{11}$$

Then $[L^2, z] = 2i\hbar(xL_y - yL_x - i\hbar z)$.

$$\begin{aligned} [L^2, [L^2, z]] &= 2i\hbar\{[L^2, x]L_y + x\cancelto{0}{[L^2, L_y]} - [L^2, y]L_x - y\cancelto{0}{[L^2, L_x]} - i\hbar[L^2, z]\} \\ &= -2\hbar^2\{2(yL_z - zL_y - i\hbar x)L_y - 2(zL_x - xL_z - i\hbar y)L_x - (L^2 z - zL^2)\} \\ &= -2\hbar^2 \left\{ 2yL_zL_y \underbrace{-2zL_y^2 - 2zL_x^2}_{-2zL^2+2zL_z^2} -2i\hbar xL_y + 2xL_zL_x + 2i\hbar yL_x - L^2 z + zL^2 \right\} \\ &= 2\hbar^2(zL^2 + L^2 z) - 4\hbar^2 \left\{(yL_z - i\hbar x)L_y + zL_z^2 + (xL_z + i\hbar y)L_x\right\} \\ &= 2\hbar^2(zL^2 + L^2 z) - 4\hbar^2 \underbrace{(L_z yL_y + zL_z^2 + L_z xL_x)}_{L_z(\mathbf{r}\cdot\mathbf{L})=0} \end{aligned} \tag{12}$$

Problem 9.13 (3 points)

The integral under consideration is

$$\langle n'00|\mathbf{r}|n00\rangle = \frac{1}{4\pi}\int dxdydz\, R_{n'0}(r)R_{n0}(r)(x\hat{x} + y\hat{y} + z\hat{z}). \tag{13}$$

$R_{n'0}(r)$ and $R_{n0}(r)$ are even in x, y and z, so the integrand is odd with respect to those variables and the integral is zero.

Problem 9.15 (15 points)

(a) Using $|\psi(t)\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle$ in the Schrödinger equation with $H = H_0 + H'(t)$ yields

$$\cancel{\sum_n c_n e^{-iE_nt/\hbar}E_n|n\rangle} + \sum_n c_n e^{-iE_nt/\hbar}H'|n\rangle = i\hbar\sum_n \dot{c}_n e^{-iE_nt/\hbar}|n\rangle + \cancel{i\hbar\left(-\frac{i}{\hbar}\right)\sum_n c_n e^{-iE_nt/\hbar}E_n|n\rangle} \tag{14}$$

Multiplying both sides by $\langle m|$, using that $\langle m|n\rangle = \delta_{mn}$ and defining $H'_{mn} = \langle m|H'|n\rangle$ results in

$$\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n e^{i(E_m - E_n)t/\hbar}H'_{mn} \tag{15}$$

(b) Similar to the two-level case in Griffiths 9.1.2, the zeroth order terms are $c_N(t) = 1$ and $c_m(t) = 0$ for $m \neq N$. We insert these values in Eq. (15)

$$\dot{c}_N = -\frac{i}{\hbar} H'_{NN} \implies c_N(t) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H'_{NN}(t') \tag{16}$$

and

$$\dot{c}_m = -\frac{i}{\hbar} e^{i(E_m - E_N)t/\hbar} H'_{mN} \implies c_m(t) = -\frac{i}{\hbar}\int_0^t dt'\, e^{i(E_m - E_N)t'/\hbar} H'_{mN}(t') \tag{17}$$

(c) From Eq. (17)

$$\begin{aligned} c_M(t) &= -\frac{i}{\hbar} H'_{MN} \int_0^t dt'\, e^{i(E_M - E_N)t'/\hbar} \\ &= -\frac{2iH'_{MN}}{E_M - E_N} e^{i(E_M - E_N)t/2\hbar} \sin\left(\frac{E_M - E_N}{2\hbar} t\right) \end{aligned} \tag{18}$$

Therefore,

$$P_{N\to M} = \frac{4|H'_{MN}|^2}{(E_M - E_N)^2} \sin^2\left(\frac{E_M - E_N}{2\hbar} t\right) \tag{19}$$

(d)

$$\begin{aligned} c_M(t) &= -\frac{i}{2\hbar} V'_{MN} \int_0^t dt'\, e^{i(E_M - E_N)t'/\hbar} \left(e^{i\omega t'} + e^{-i\omega t'}\right) \\ &= -\frac{i}{2\hbar} V'_{MN} \left[\frac{e^{i(\hbar\omega + E_M - E_N)t'/\hbar}}{i(\hbar\omega + E_M - E_N)/\hbar} + \frac{e^{i(-\hbar\omega + E_M - E_N)t'/\hbar}}{i(-\hbar\omega + E_M - E_N)/\hbar}\right]_0^t \end{aligned} \tag{20}$$

If $E_M < E_N$ ($E_M > E_N$), the first (second) term dominates and transitions occur only for $\omega \approx \mp(E_M - E_N)/\hbar$. Therefore,

$$P_{N\to M} = \frac{|V_{MN}|^2}{(E_M - E_N \pm \hbar\omega)^2} \sin^2\left(\frac{E_M - E_N \pm \hbar\omega}{2\hbar} t\right) \tag{21}$$

(e) For light, we consider $V_{MN} = e\langle M|\mathbf{r} \cdot E_0 \hat{\epsilon}|N\rangle$. Following Griffiths 9.2.3 we obtain

$$R_{N\to M} = \frac{\pi}{3\epsilon_0 \hbar^2} |\langle M|r|N\rangle|^2 \rho\left(\pm\frac{E_m - E_N}{\hbar}\right) \tag{22}$$

where $+(-)$ is for absorption (stimulated emission).

Problem 9.18 (6 points)

For a particle in an infinite square well potential, $E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$ and so $E_2 - E_1 = \frac{3\pi^2\hbar^2}{2ma^2}$. Using Eq. (19) and Mathematica for the numerical evaluation we obtain

$$H'_{12} = \frac{2}{a}\int_0^{a/2} dx\, \sin\left(\frac{\pi x}{a}\right) V_0 \sin\left(\frac{2\pi x}{a}\right) = \frac{4V_0}{3\pi}. \tag{23}$$

Therefore

$$\begin{aligned} P_{1\to 2} &= 4\left(\frac{4V_0}{3\pi}\right)^2 \left(\frac{2ma^2}{3\pi^2\hbar^2}\right)^2 \sin^2\left(\frac{3\pi^2\hbar T}{4ma^2}\right) \\ &= \left[\frac{16ma^2V_0}{9\pi^3\hbar^2} \sin\left(\frac{3\pi^2\hbar T}{4ma^2}\right)\right]^2. \end{aligned} \tag{24}$$

Problem 2 (6 points)

$\mathbf{E} = \mathcal{E}_0(\hat{x} + \hat{y} + \hat{z})e^{-t/\tau}$. The perturbation Hamiltonian is $H' = e\mathbf{r}\cdot\mathbf{E} = e\mathcal{E}_0(x+y+z)e^{-t/\tau}$. The initial state is $|i\rangle = |100\rangle$. The allowed final states are $|f\rangle = |210\rangle$, $|21\pm1\rangle$. Because of dipole selection rule $\Delta l = \pm 1$, the final state $|200\rangle$ is not allowed. From Eq. (17)

$$
\begin{aligned}
|c_{i\to f}(t\to\infty)|^2 &= \left(\frac{e\mathcal{E}_0}{\hbar}\right)^2 |\langle f|(x+y+z)|i\rangle|^2 \left|\int_0^\infty dt'\, e^{(i\omega_0 - 1/\tau)t'}\right|^2 \\
&= \left(\frac{e\mathcal{E}_0}{\hbar}\right)^2 |\langle f|(x+y+z)|i\rangle|^2 \frac{1}{1/\tau^2 + \omega_0^2}
\end{aligned}
\tag{25}
$$

where $\omega_0 = (E_2 - E_1)/\hbar$ and $E_n = -13.6/n^2$ eV. The dipole matrix element can be easily calculated using Mathematica with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. The wavefunctions for hydrogen $\psi_{nlm}(r,\theta,\phi)$ are given by

$$
\text{In[6]:=}\ \psi[n\_, l\_, m\_, r\_, \theta\_, \phi\_] := \sqrt{\left(\frac{2}{n\,a}\right)^3 \frac{(n-l-1)!}{2\,n\,((n+l)!)}}\ \text{Exp}\left[\frac{-r}{n\,a}\right] \left(\frac{2\,r}{n\,a}\right)^l\ \text{LaguerreL}\left[n-l-1,\ 2\,l+1,\ \frac{2\,r}{n\,a}\right]\ \text{SphericalHarmonicY}[l, m, \theta, \phi]
$$

In this way,

$$
\begin{aligned}
\langle 210|x+y+z|100\rangle &= \frac{2^7\sqrt{2}a_0}{3^5} \\
\langle 21\pm1|x+y+z|100\rangle &= \frac{2^7 a_0}{3^5}(\mp 1 - i)
\end{aligned}
\tag{26}
$$

where a_0 is the Bohr radius. The transition probabilities are therefore

$$
|c_{100\to210}|^2 = |c_{100\to21\pm1}|^2 = \frac{2^{15}}{3^{10}}\left(\frac{e\mathcal{E}_0 a_0}{\hbar}\right)^2 \frac{1}{1/\tau^2 + \omega_0^2}
\tag{27}
$$